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TOR GIVES THE INVERSE TO THE HILBERT FUNCTION OF A GRADED ALGEBRA

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Dedicated to Jack Kiefer: I still have the feeling of sharing with him

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The Hilbert generating function of $\mathrm{Tor}^U(A, A)$ gives the inverse of the Hilbert generating function of U where U is a graded A algebra.

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1. Introduction

Suppose k is a field, $U=\bigoplus_{i=0}^{\infty} U_i$ is a graded vector space over k with $\dim U_i<\infty$ and $\Psi(U)(t)=\sum_{i=0}^{\infty}(\dim U_i)t^i\in\mathbb{Z}[[t]]$. This is the Hilbert (generating) function of U and we shall describe $[\Psi(U)(t)]^{-1}$ when U is a graded k-algebra with $U_0=k$.

For a vector space V over k let $TV, \Lambda V$ and SV denote the tensor, exterior and symmetric algebras on V. Let DV denote $SV/\langle\{vw\}_{v,w\in V}\rangle$. With their usual grading one can verify

$$[\Psi(DV)(t)][\Psi(TV)(-t)]=1, \tag{1.1}$$

$$[\Psi(SV)(t)][\Psi(\Lambda V)(-t)]=1. \tag{1.2}$$

The appropriate K-theoretic generalization of (1.2) appears in [1, p. 528, (8.4)]. This beautiful and mysterious result lead to the present paper.

In both cases (1.1) and (1.2) there are natural graded vector space isomorphisms

$$TV\cong\mathrm{Tor}^{DV}(k,k), \qquad \Lambda V\cong\mathrm{Tor}^{SV}(k,k), \tag{1.3}$$

which suggests the following conjecture:

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Suppose U is a graded k algebra with $U_0 = k$. Then

$$[\Psi(U)(t)][\Psi(\mathrm{Tor}^U(k,k))(-t)] = 1. \tag{1.4}$$

As it stands the conjecture is false but on the right track. A bigrading on $\mathrm{Tor}^U(k,k)$ comes into the picture. For a suitably bigraded vector space M

$$\Phi(M) = \sum_{j=0}^{\infty}\left(\sum_{i=0}^{\infty}(-)^i \dim M_{ij}\right)t^j. \tag{1.5}$$

'Suitably' refers to the fact that for fixed $j, M_{ij} = \{0\}$ for large i. The main theorem (2.10) of this paper is that (1.4) is true with $\Psi(\mathrm{Tor}^U(k,k))(-t)$ replaced by $\Phi(\mathrm{Tor}^U(k,k))$. The reason that (1.1) and (1.2) hold with Ψ instead of Φ is that the bigrading on Tor in those cases is concentrated on the diagonal. This is also why the '$-t$' appears there but not with Φ.

We state the (corrected) (1.4) in Theorem 2.10 and prove it at 3.12. U need not be commutative and we work in the appropriate K-theoretic setting without presuming any knowledge of K-theory. The needed ideas are developed in the first few paragraphs of Section 2.

A final word about computing Φ. Notice in (1.5) the inner sum is an alternating sum of dimensions. Given a finite complex of finite dimensional vector spaces, the alternating sum of dimensions is a homology invariant. This is the key to computing $\Phi(M)$, Proposition 3.11. At two points in the proof of the main theorem we put a differential on a bigraded module M and replace $\Phi(M)$ by $\Phi(H(M))$. Then we *recognize* $H(M)$. In the first case $H(M) = \mathrm{Tor}^U(k,k)$ and in the second case – for a different$M$ – $H(M)$ collapses.

2. Preliminaries and statement of the theorem

For the duration A and S are commutative rings, $\mathcal{M}$ is the class of finitely generated projective A-modules and for each $M \in \mathcal{M}$ there is $[M] \in S$ where [ ] has the following properties for $A, M', M, M'' \in \mathcal{M}$:

(2.1.i) $[A] = 1_S$.

(2.1.ii) $[M] = [M'] + [M'']$ if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules.

(2.1.iii) $[M][M'] = [M \otimes M']$.

It is easy to verify that for $\{0\}, M, M' \in \mathcal{M}$:

(2.2.iv) $[\{0\}] = 0_S$ where $\{0\}$ is the zero module.

(2.2.v) $[M] = [M']$ if $M \cong M'$ as A-modules.

(2.2.vi) $[M \oplus M'] = [M] \oplus [M']$.

The *universal* such S for a given A is $K_0(A)$. This defines $K_0(A)$.

The classical example of such $A, S, \mathcal{M}$ and [ ] is where A is a field, $\mathcal{M}$ is 'finite dimensional vector spaces over A', $S = \mathbb{Z}$ and $[M] = \dim_A M$.

We use $|\mathbb{Z}|$ to denote the set $\{0, 1, 2, \dots\}$.

2.3. Definition. We let $\mathcal{M}^{|\mathbb{Z}|}$ denote the class of $|\mathbb{Z}|$-graded projective A-modules of the form $U = \bigoplus_{i\in |\mathbb{Z}|} U_i$ where for each i, $U_i \in \mathcal{M}$. For such U the grading is part of the structure of U and we can define

$$\Psi(U) = \sum_{k=0}^{\infty} [U_k]t^k \in S[[t]]. \tag{2.4}$$

If $U_0 \cong A$ as A-modules then $\Psi(U) = 1 + \sum_{i=1}^{\infty} s_i t^i$, an *invertible* power series in $S[[t]]$. The purpose of this paper is to find a V arising naturally from U where $\Psi(V) = \Psi(U)^{-1}$. For this purpose we shall have to extend the definition of Ψ to bigraded modules. Notice that if $U, V \in \mathcal{M}^{|\mathbb{Z}|}$ and if $U \otimes V$ has the usual graded tensor product structure then $U \otimes V \in \mathcal{M}^{|\mathbb{Z}|}$ and $\Psi(U \otimes V) = \Psi(U)\Psi(V)$. This is an easy consequence of (i)–(vi).

2.5. Definition. A *row finite* $|\mathbb{Z}|$-bigraded A-module is an A-module of the form $C = \bigoplus_{0 \le i,k \in \mathbb{Z}} C_{ik}$ where for fixed k: $C_{ik} = \{0\}$ for large i. $\mathcal{M}^{|\mathbb{Z}|^2}$ denotes the class of row finite $|\mathbb{Z}|$-bigraded A-modules C where $C_{ik} \in \mathcal{M}$ for each i and k.

For $C \in \mathcal{M}^{|\mathbb{Z}|^2}$ define

$$\Phi(C) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{\infty} (-)^i [C_{ik}] \right) t^k \in S[[t]]. \tag{2.6}$$

The inner sum is finite since for fixed k: C_{ik} is zero for large i.

For $|\mathbb{Z}|$-bigraded A-modules C and D let

$$(C \otimes D)_{nm} = \bigoplus_{\substack{i+j=n \\ k+l=m}} C_{ik} \otimes D_{jl} \tag{2.7}$$

and

$$C \otimes D = \bigoplus_{m,n \in |\mathbb{Z}|} (C \otimes D)_{nm}. \tag{2.8}$$

This defines the $|\mathbb{Z}|$-bigraded A-module structure on $C \otimes D$. It is easy to check that $C \otimes D$ is row finite if both C and D are. Also, that $C \otimes D \in \mathcal{M}^{|\mathbb{Z}|^2}$ if $C, D \in \mathcal{M}^{|\mathbb{Z}|^2}$.

2.9. Lemma. *Suppose $U \in \mathcal{M}^{|\mathbb{Z}|}$ and $C, D \in \mathcal{M}^{|\mathbb{Z}|^2}$.*

(a) *If U' is the $|\mathbb{Z}|$-bigraded A-module defined by*

$$U'_{ik} = \begin{cases} U_k & \text{for } i = 0, \\ \{0\} & \text{otherwise} \end{cases}$$

then $U' \in \mathcal{M}^{|\mathbb{Z}|^2}$ and $\Phi(U') = \Psi(U)$.

(b) $\Phi(C)\Phi(D) = \Phi(C \otimes D)$.

Proof. (a) This is easy and left to you. (b) Calculate

$$\Phi(C)\Phi(D) = \left(\sum_{k=0}^{\infty} \sum_{i=0}^{\infty} (-)^i [C_{ik}] t^k \right) \left(\sum_{l=0}^{\infty} \sum_{j=0}^{\infty} (-)^j [D_{jl}] t^l \right)$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-)^{i+j} \underbrace{[C_{ik}][D_{jl}]}_{[C_{ik} \otimes D_{jl}]} t^{k+l}$$

$$= \sum_{k+l=m} \sum_{i+j=n} (-)^n [(C \otimes D)_{nm}] t^m = \Phi(C \otimes D). \qquad \square$$

The element of $\mathcal{M}^{|\mathbb{Z}|^2}$ to which we must apply Φ is a certain Tor. Usually Tor is only singly graded. The bigrading on Tor comes from the bar resolution used to calculate Tor for a graded algebra. This may be taken as an ad hoc definition of the bigrading on Tor but the bigrading is *natural* in the category of graded modules for graded algebras. Here are a few words to that point.

If U is a $|\mathbb{Z}|$-graded A-algebra (i.e. an A-algebra and $|\mathbb{Z}|$-graded A-module where $U_i U_j \subset U_{i+j}$), then a left $|\mathbb{Z}|$-graded U-module M is simultaneously a left U-module and $|\mathbb{Z}|$-graded A-module where $U_i M_j \in M_{i+j}$. $\mathcal{K}$ is the category of left $|\mathbb{Z}|$-graded U-modules where the morphisms are degree preserving graded U-module maps. Considering *U with its degrees shifted* shows that K has enough projectives to form projective resolutions. If L is a $|\mathbb{Z}|$-graded right U-module then with the usual grading on tensor product $L \otimes_U M$ is a $|\mathbb{Z}|$-graded A-module. Hence $L \otimes_U -$ is an additive functor from $\mathcal{K}$ to the category of $|\mathbb{Z}|$-graded A-modules and we may take its derived functors. Since a projective resolution of M in $\mathcal{K}$ is also a projective resolution of M as a U-module with the grading ignored the derived functors of $L \otimes_U -$ applied to $M \subset \mathcal{K}$ coincide with ordinary $\operatorname{Tor}_*^U(L,M)$. Again considering a projective resolution of M in $\mathcal{K}$, $\operatorname{Tor}_n^U(L,M)$, the nth derived functor of $L \otimes_U -$ will be a $|\mathbb{Z}|$-graded A-module. Thus for $n, m \in |\mathbb{Z}|$ we have $\operatorname{Tor}_n^U(L,M)_m$ and this gives the $|\mathbb{Z}|$-bigrading on $\operatorname{Tor}^U(L,M)$. This $|\mathbb{Z}|$-bigrading on $\operatorname{Tor}^U(L,M)$ arises in computing $\operatorname{Tor}^U(L,M)$ from the bar resolution since this is a resolution of M in $\mathcal{K}$. Henceforth we shall work with the $|\mathbb{Z}|$-bigrading on $\operatorname{Tor}^U(L,M)$ as it arises from the bar resolution.

The main result in this paper is:

2.10. Theorem. *Suppose U is a $|\mathbb{Z}|$-graded A-algebra where $U_0 \cong A$ and $U \in \mathcal{M}^{|\mathbb{Z}|}$. Let A have the trivial graded left and right U-module structures where $A = A_0$ and $U_i A = 0 = A U_i$ for $0 < i \in \mathbb{Z}$. If $\operatorname{Tor}^U(A,A)$ is a projective A-module then $\operatorname{Tor}^U(A,A) \in \mathcal{M}^{|\mathbb{Z}|^2}$ and $1 = \Psi(U)\Phi(\operatorname{Tor}^U(A,A))$ in $S[[t]]$.*

The next section will be devoted to the proof of this theorem. First some consequences.

The theorem shows that $\Phi(\operatorname{Tor}^U(A,A))$ only depends on the A-module structure of U since inverses are unique. Written down carefully this becomes:

2.11. Corollary. *Suppose U and V are $|\mathbb{Z}|$-graded A-algebras where $U_0 = A = V_0$ and $U, V \in \mathcal{M}^{|\mathbb{Z}|}$. Let A have the trivial graded left and right U and V-module structures as in the theorem. Then $\Phi(\operatorname{Tor}^U(A,A)) = \Phi(\operatorname{Tor}^V(A,A))$ if $U \cong V$ as* *graded A-modules and both* $\mathrm{Tor}^U(A,A)$ *and* $\mathrm{Tor}^V(A,A)$ *are projective A-modules.*

As the corollary shows the algebra structure on U does not affect $\Phi(\mathrm{Tor}^U(A,A))$. Can we get away with no algebra structure on U? Suppose $U\in\mathcal{M}^{\mathbb{Z}}$ where $U_0\cong A$ as an A-module. Then there is a (unique) A-algebra structure on U where $U_0=A$ and $U_iU_j=\{0\}$ for $i,j\in\mathbb{N}=\{1,2,3,\dots\}$. With this trivial A-algebra structure on U we can form $\mathrm{Tor}^U(A,A)$. As a corollary (3.24) to our proof of Theorem 2.10 we shall see that for this algebra structure on U, $\mathrm{Tor}^U(A,A)$ is a projective A-module. Hence Theorem 2.10 applies and $\Phi(\mathrm{Tor}^U(A,A))$ gives the inverse to $\Psi(U)$. In Corollary 3.24 we explicitly give $\mathrm{Tor}^U(A,A)$ and $\Phi(\mathrm{Tor}^U(A,A))$.

3. Proof of the theorem

At first we collect results needed for the proof of the main theorem, then prove it. The next definition will make it easy to work with finite complexes without tacking on initial or final zeros and will avoid keeping track of degrees.

3.1. Definition. Suppose

$$\cdots\to M\to N\to P\to\cdots$$

is a sequence of A-modules and module maps. If X is one of the modules we say that the sequence is a *complex at* X and define $H(X)$ as follows:

(a) If the sequence is only the module X then it is a complex at X and $H(X)=X$.

(b) If X is a *left end*, i.e. the sequence begins

$$X\xrightarrow{\varepsilon}Y\quad\text{(and possibly continues from } Y),$$

then it is a complex at X and $H(X)\cong\operatorname{Ker}\varepsilon$, a submodule.

(c) If X is a *right end*, i.e. the sequence ends

$$\text{(possibly leading up to } W \text{ and ending)}\quad W\xrightarrow{\partial}X,$$

then it is a complex at X and $H(X)\cong\operatorname{Coker}\partial$, a quotient module.

(d) If X is *internal*, i.e. the sequence contains

$$\text{(possibly leading up)}\quad W\xrightarrow{\partial}X\xrightarrow{\varepsilon}Y\quad\text{(possibly continuing)},$$

then it is a complex at X if $\varepsilon\partial=0$; i.e. $\operatorname{Ker}\varepsilon\supset\operatorname{Im}\partial$ and $H(X)\cong\operatorname{Im}\partial/\operatorname{Ker}\varepsilon$, a subquotient module.

The sequence is a *complex* if it is a complex at each of its modules. In this case the module $H(X)$ is defined for each module X in the sequence.

3.2. Lemma. *Suppose that*

$$M \xrightarrow{\delta} N_t \longrightarrow \cdots \longrightarrow N_0$$

is a complex where M and $\{N_i\}_0^t$ and $\{H(N_i)\}_0^t$ are projective A-modules. Then:

(a) $\operatorname{Im}\delta$ *and* $\operatorname{Ker}\delta = H(M)$ *are projective A-modules.*

(b) *If in addition M and $\{N_i\}$ are finitely generated A-modules then* $\operatorname{Im}\delta$, $\operatorname{Ker}\delta = H(M)$ *and $\{H(N_i)\}_0^t$ are finitely generated A-modules.*

With S and [ ] as in (2.1),

$$(-)^{t+1}[M] + \sum_{i=0}^{t} (-)^i [N_i] = (-)^{t+1}[H(M)] + \sum_{i=0}^{t} (-)^i [H(N_i)] \tag{3.3}$$

Proof. We proceed by induction on t. The case $t=-1$ is not excluded. Here the complex is simply M and the assertions are trivially true.

For $t=0$ we have $M \xrightarrow{\delta} N_0$ giving the exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Im}\delta \arrow[r] & N_0 \arrow[r] & \operatorname{Coker}\delta \arrow[r] \arrow[d, equal] & 0 \\ & & & H(N_0) & \end{tikzcd} \tag{3.4}$$

$$\begin{tikzcd} & H(M) \arrow[d, equal] & & & \\ 0 \arrow[r] & \operatorname{Ker}\delta \arrow[r] & M \arrow[r] & \operatorname{Im}\delta \arrow[r] & 0 \end{tikzcd} \tag{3.5}$$

Projectivity of $H(N_0)$ implies that (3.4) is split. Hence N_0 being projective implies that $\operatorname{Im}\delta$ is projective. N_0 being finitely generated implies finite generation of $\operatorname{Coker}\delta = H(N_0)$ just as M being finitely generated implies finite generation of $\operatorname{Im}\delta$.

Projectivity of $\operatorname{Im}\delta$ implies that (3.5) is split. Hence M being projective implies that $H(M)$ is projective and M being finitely generated implies that $H(M)$ is finitely generated.

By (2.1.ii), (3.4) and (3.5) give

$$[N_0] = [H(N_0)] + [\operatorname{Im}\delta], \qquad [M] = [H(M)] + [\operatorname{Im}\delta]. \tag{3.6}$$

Subtracting gives (3.3) for $t=0$.

Now suppose we have the complex

$$M \xrightarrow{\delta} N_t \xrightarrow{\varepsilon} \cdots \longrightarrow N_0$$

for $t \geq 1$ and the results have been proved for smaller t. Consider the complex

$$\begin{tikzcd} M' \arrow[r, "\varepsilon"] \arrow[d, equal] & N'_{t-1} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & N'_0 \arrow[d, equal] \\ N_t & N_{t-1} & & N_0 \end{tikzcd} \tag{3.7}$$

Then $H(N_i')=H(N_i)$ for $i=0,\dots,t-1$ and so is projective by hypothesis. Also M' and $\{N_i'\}_{i=0}^{t-1}$ are projective and by the induction $\operatorname{Im}\varepsilon$ and $\operatorname{Ker}\varepsilon=H(M')$ are projective. Also if $\{N_i\}_0^t$ are finitely generated then so are $\operatorname{Im}\varepsilon$, $\operatorname{Ker}\varepsilon=H(M')$, $\{H(N_i')=H(N_i)\}_{i=0}^{t-1}$ and

$$\begin{array}{ccc} (-)^t[M']+\sum_{i=0}^{t-1}(-)^i[N_i'] & = & (-)^t[H(M')]+\sum_{i=0}^{t-1}(-)^i[H(N_i')] \\ \| & & \| \\ \sum_{i=0}^{t}(-)^i[N_i] & & (-)^t[\operatorname{Ker}\varepsilon]+\sum_{i=0}^{t-1}(-)^i[H(N_i)] \end{array} \tag{3.8}$$

Consider $M\xrightarrow{\delta}\operatorname{Ker}\varepsilon$, since $\operatorname{Im}\delta\subset\operatorname{Ker}\varepsilon$. By what we have shown for $t=0$ it follows that $\operatorname{Im}\delta$ and $\operatorname{Ker}\delta=H(M)$ are projective. This uses the fact that $H(\operatorname{Ker}\varepsilon)=\operatorname{Ker}\varepsilon/\operatorname{Im}\delta=H(N_t)$ is assumed to be projective. If M and $\operatorname{Ker}\varepsilon$ are finitely generated then by the case $t=0$: $\operatorname{Im}\delta$, $\operatorname{Ker}\delta=H(M)$ and $H(\operatorname{Ker}\varepsilon)=H(N_t)$ are finitely generated and

$$\begin{array}{c} [\operatorname{Ker}\varepsilon]-M=[H(\operatorname{Ker}\varepsilon)]-[H(M)] \\ \| \\ [H(N_t)] \end{array}$$

Hence

$$[M]=[H(M)]-[H(N_t)]+[\operatorname{Ker}\varepsilon].$$

Adding this to or subtracting this from (3.8) gives (3.3) and completes the induction. □

3.9. Corollary. *Under the hypothesis of* (3.2) *for* $i=0,\dots,t-1$ *the maps* $N_{i+1}\to N_i$ *have projective image and kernel which are finitely generated if* $\{N_i\}_0^t$ *are finitely generated.*

Proof. Apply the lemma to the complex $N_{i+1}\to N_i\to\cdots\to N_0$. □

3.10. Definition. Suppose $C\in\mathcal{M}^{|\mathbb{Z}|^2}$ (Definition 2.5). A *sleeve job* on C is an A-module map $d:C\to C$ of degree $(-1,0)$ where $d^2=0$ and the resulting homology is a projective A-module. Note, $C=\bigoplus_{i,k\in|\mathbb{Z}|}C_{ik}$ and saying that d is of degree $(-1,0)$ means that

$$d(C_{ik})\begin{cases}=\{0\} & \text{for } i=0,\\ \subset C_{i-1,k} & \text{for } i\in|\mathbb{Z}|.\end{cases}$$

The picture to keep in mind is that of Diagram 1.

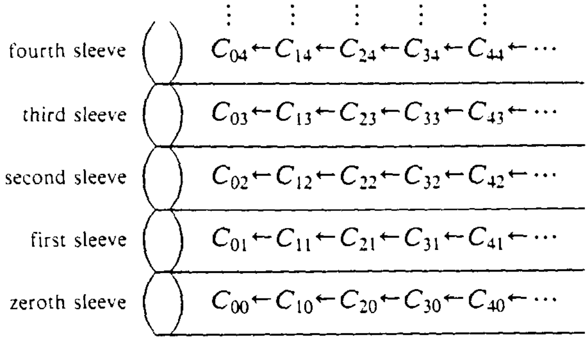

Diagram 1

Since the homology with respect to d is assumed to be a projective A-module the individual summands $H(C_{ik})$ are projective A-modules. The C_{ik} are finitely generated projective A-modules; hence, by Lemma 3.2 each $H(C_{ik}) \in \mathcal{M}$. Thus if $H(C)$ denotes the $|\mathbb{Z}|$-bigraded A-module $\bigoplus_{i,k \in \mathbb{Z}} H(C_{ik})$ then $H(C) \in \mathcal{M}^{|\mathbb{Z}|^2}$.

3.11. Proposition. *Suppose $C \in \mathcal{M}^{|\mathbb{Z}|^2}$ and d is a sleeve job on C. Then*

$$\Phi(C) = \Phi(H(C)).$$

Proof. This is immediate from (3.3) and the definition of Φ in (2.6). □

3.12. Proof of 2.10. Let C be the $|\mathbb{Z}|$-bigraded module with U as its zeroth column and zero elsewhere. That is

$$C_{ik} = \begin{cases} U_k & \text{for } i = 0, \\ \{0\} & \text{otherwise.} \end{cases} \tag{3.13}$$

By Lemma 2.9(a), $C \in \mathcal{M}^{|\mathbb{Z}|^2}$ and $\Phi(C) = \Psi(U)$. Hence it suffices to prove that $1 = \Phi(C)\Phi(\mathrm{Tor}^U(A, A))$ in $S[[t]]$.

Define D as the $|\mathbb{Z}|$-bigraded module where

$$D_{nl} = \bigoplus_{\substack{j_1 + \cdots + j_n = l \\ j_1, \ldots, j_n \in \mathbb{N}}} U_0 \otimes U_{j_1} \otimes \cdots \otimes U_{j_n} \otimes U_0 \tag{3.14}$$

where $\mathbb{N}$ denotes the set $\{1, 2, 3, \ldots\}$.

The lower left corner of D is given in Diagram 2.

Row 4	$\{0\}$	$U_0 \otimes U_4 \otimes U_0$	$U_0 \otimes U_1 \otimes U_3 \otimes U_0$ $\oplus$ $U_0 \otimes U_2 \otimes U_2 \otimes U_0$ $\oplus$ $U_0 \otimes U_3 \otimes U_1 \otimes U_0$	$U_0 \otimes U_1 \otimes U_1 \otimes U_2 \otimes U_0$ $\oplus$ $U_0 \otimes U_1 \otimes U_2 \otimes U_1 \otimes U_0$ $\oplus$ $U_0 \otimes U_2 \otimes U_1 \otimes U_1 \otimes U_0$
Row 3	$\{0\}$	$U_0 \otimes U_3 \otimes U_0$	$U_0 \otimes U_1 \otimes U_2 \otimes U_0$ $\oplus$ $U_0 \otimes U_2 \otimes U_1 \otimes U_0$	$U_0 \otimes U_1 \otimes U_1 \otimes U_1 \otimes U_0$
Row 2	$\{0\}$	$U_0 \otimes U_2 \otimes U_0$	$U_0 \otimes U_1 \otimes U_1 \otimes U_0$	$\{0\}$
Row 1	$\{0\}$	$U_0 \otimes U_1 \otimes U_0$	$\{0\}$	$\{0\}$
Row 0	$U_0 \otimes U_0$	$\{0\}$	$\{0\}$	$\{0\}$
	Column 0	Column 1	Column 2	Column 3

Diagram 2

If $U^+ = U_1 \oplus U_2 \oplus U_3 \oplus \cdots$ and K_n denotes the direct sum of the terms in the nth column for $n \in |\mathbb{Z}|$, then

$$K_n = U_0 \otimes \underbrace{U^+ \otimes \cdots \otimes U^+}_{n\text{-times}} \otimes U_0 \tag{3.15}$$

as a graded A-module, where K_n is graded by row and $U_0 \otimes \bigotimes^n U^+ \otimes U_0$ has the usual grading on a tensor product.

Identify U^+ with U/U_0 via the composite $U^+ \hookrightarrow U \to U/U_0$. Consider the bar resolution of U_0 as a left U-module described in [2, pp. 280–283]. The n-th term is $B_n(U, U_0)$ in the notation of [2] and under the identification of U/U_0 and U^+

$$B_n(U, U_0) = U \otimes \underbrace{U^+ \otimes \cdots \otimes U^+}_{n\text{-times}} \otimes U_0. \tag{3.16}$$

The boundary map [2, p. 281, (2.5)] is determined by

$$\begin{array}{ccc} B_n(U, U_0) & & u_0 \otimes u_1 \otimes \cdots \otimes u_n \otimes \beta \\ \downarrow & & \downarrow \\ B_{n-1}(U, U_0) & & \sum_{i=1}^{n} (-)^i u_0 \otimes \cdots \otimes u_{i-2} \otimes (u_{i-1} u_i) \otimes u_{i+1} \otimes \cdots \otimes u_n \otimes \beta \end{array} \tag{3.17}$$

for $u_0 \in U$, $u_1, \ldots, u_n \in U^+$, $\beta \in A = U_0$. In the sum at (3.17) the expected final term

$u_0\otimes\cdots\otimes u_{n-1}\otimes(u_n\lambda)$ is ostensibly missing. This is because the '$u_n\lambda$' is the module action of u_n on $\lambda\in A$ and not product in U of u_n by λ. Since $u_n\in U^+$ the module action '$u_n\lambda$' is zero. Since U is a projective A-module the bar resolution gives a projective resolution of $U_0=A$. The explicit description of the boundary map at (3.17) shows that the bar resolution gives a projective resolution of U_0 in the category of graded left U-modules if $B_n(U,U_0)$ has the usual grading on tensor products. Now consider U_0 as a graded right U-module and apply the functor $U_0\otimes_U -$ to the bar resolution to compute $\mathrm{Tor}^U(U_0,U_0)=\mathrm{Tor}^U(A,A)$ in the category of graded U-modules. The nth term of the complex is $U_0\otimes_U B_n(U,U_0)$ which by (3.16) and (3.15) is K_n. From (3.17) the boundary map on the K_n's is determined by

$$\begin{array}{ccc} K_n & & \alpha\otimes u_1\otimes\cdots\otimes u_n\otimes\beta \\ \downarrow & & \downarrow \\ K_{n-1} & & \sum_{i=2}^{n}(-)^i\alpha\otimes u_1\otimes\cdots\otimes u_{i-2}\otimes(u_{i-1}u_i)\otimes u_{i+1}\otimes\cdots\otimes u_n\otimes\beta \end{array} \tag{3.18}$$

for $\alpha,\beta\in A=U_0$, $u_1,\ldots,u_n\in U^+$. In (3.18) the expected first term $(\alpha u_1)\otimes u_2\otimes\cdots\otimes u_n\otimes\beta$ is ostensibly missing for the reason given below (3.17). And for this same reason the map at (3.18) stands for the zero map when $n=1$.

This boundary map between the K_n's carries D_{nl} to $D_{n-1,l}$, hence gives a sleeve job on D. Since the homology of the complex $\{K_n\}_0^\infty$ is $\mathrm{Tor}^U(A,A)$, we see by Theorem 3.11 that

$$\Phi(D)=\Phi(\mathrm{Tor}^U(A,A)). \tag{3.19}$$

Next consider $C\otimes D$ with the bigrading described in (2.7). Then

$$(C\otimes D)_{nl}=\bigoplus_{\substack{j_0+\cdots+j_n=l\\ j_0\in\mathbb{Z}\\ j_1,\ldots,j_n\in\mathbb{N}}} U_{j_0}\otimes\cdots\otimes U_{j_n}\otimes U_0. \tag{3.20}$$

For $C\otimes D$ the nth column $-\bigoplus_{i\in\mathbb{Z}}(C\otimes D)_{ni}-$ is

$$U\otimes\underbrace{U^+\otimes\cdots\otimes U^+}_{n\text{-times}}\otimes U_0 \tag{3.21}$$

which we recognize as $B_n(U,U_0)$. The boundary map of the bar resolution – given explicitly at (3.17) – gives a sleeve job on $C\otimes D$. Hence by Theorem 3.11

$$\Phi(H(C\otimes D))=\Phi(C\otimes D). \tag{3.22}$$

The homology of this complex $C\otimes D$ is the homology of the bar resolution of U_0, a graded projective resolution. Hence $H((C\otimes D)_{nm})=U_0$ for $n=0=m$ and is zero otherwise. From the definition of Φ at (2.6) and the fact that $U_0=A$ and $[A]=1$ we get

$$\Phi(H(C\otimes D))=1. \tag{3.23}$$

Putting together (3.23), (3.22), Lemma 2.9(b), (3.19) and Lemma 2.9(a), in the order indicated, gives

$$1 = \Phi(H(C \otimes D)) = \Phi(C \otimes D) = \Phi(C)\Phi(D)$$
$$= \Phi(C)\Phi(\mathrm{Tor}^U(A, A)) = \psi(U)\Phi(\mathrm{Tor}^U(A, A)). \qquad \square$$

3.24. Corollary to 3.12. *Suppose $U \in \mathcal{U}^{|\mathbb{Z}|}$ is an A-algebra where $U_0 = A$ and $U^+U^+ = \{0\}$. Then* $\mathrm{Tor}^u(A, A)$ *as a $|\mathbb{Z}|$-bigraded module is given by*

$$\mathrm{Tor}_n^u(A, A)_l = \bigoplus_{\substack{j_1 + \cdots + j_n = l \\ j_1, \ldots, j_n \in \mathbb{N}}} U_{j_1} \otimes \cdots \otimes U_{j_n}.$$

Hence $\mathrm{Tor}^U(A, A)$ is a projective A-module and $\Phi(\mathrm{Tor}^u(A, A)) = \psi(U)^{-1}$ in $S[[t]]$. Explicitly $\Phi(\mathrm{Tor}^u(A, A))$ is given by

$$\Phi(\mathrm{Tor}^u(A, A)) = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \sum_{\substack{j_1 + \cdots + j_n = l \\ j_1, \ldots, j_n \in \mathbb{N}}} (-)^n [U_{j_1}] \cdots [U_{j_n}] t^l$$
$$= 1 - [U_1]t - ([U_2] - [U_1]^2)t^2 - ([U_3] - 2[U_1][U_2] + [U_1]^3)t^3$$
$$- ([U_4] - (2[U_1][U_3] + [U_2]^2) + 3[U_1]^2[U_2] - [U_1]^4)t^4 - \cdots.$$

Proof. As mentioned between (3.17) and (3.10) the complex $D = \bigotimes_{n \in \mathbb{Z}} K_n$ computes $\mathrm{Tor}^u(A, A)$ and the boundary map is given at (3.18). By the assumption that $U^+U^+ = \{0\}$, the boundary map at (3.18) is the zero map. Hence the homology is the complex, i.e.

$$\mathrm{Tor}^u(A, A) = H(D) = D,$$

proving the first assertion, since the U_0's in D, (3.14), can be omitted. The formula for $\Phi(\mathrm{Tor}^u(A, A))$ is the direct result of the definition of Φ in (2.6). $\square$

References

[1] H. Bass, Algebraic K-Theory (Benjamin, New York, 1968).
[2] S. MacLane, Homology (Springer, Berlin, 1963).